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ON THE LAGRANGE PROBLEM OF THE MEAN MOTION OF PERIHELIA*

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It is shown that themean motion of perihelia in the Lagrange sense in a non-resonance set is uniformly continuous in the initial phases of the frequency function.

The dynamics of a planetary system such as the solar system is examined. To a first approximation of perturbation theory, when the squares of the orbit eccentricities can be neglected compared with the eccentricities themselves, the evolution of the Laplace vector is described by the function

$$A(t) = \sum_{m=0}^{n} a_m \exp[2\pi i(\lambda_m t + \varphi_{0m})]$$

where the constants $a_m, \lambda_m, \varphi_{0m}$ are determined in terms of the planet mass and the initial conditions. The mean motion of the perihelion is defined as

$$\mu(\mathbf{a}, \boldsymbol{\lambda}, \varphi_0) = \lim_{t\to\infty} \frac{1}{t} \arg A(t)$$

Lagrange showed that $\mu = 2\pi\lambda_0$ if $a_0 \geqslant a_1 + a_2 + \ldots + a_n$. In the non-trivial case when this condition is not satisfied, the mean motion is calculated for $n = 2$ /1/ and for arbitrary n /2/ for the non-resonance set of frequencies $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$: $< k, \lambda > \neq 0$, $\forall \mathbf{k} \in Z^{n+1}$, $\mathbf{k} \neq 0$ and has the form

$$\mu(\mathbf{a}, \boldsymbol{\lambda}) = 2\pi \sum_{m=0}^{n} \lambda_m w^m(\mathbf{a})$$

$$\sum_{m=0}^{n} w^m(\mathbf{a}) = 1, \quad w^m(\mathbf{a}) \geqslant 0, \quad \mathbf{a} = (a_0, \ldots, a_n)$$

The existence of mean motion for an arbitrary set of frequencies $\boldsymbol{\lambda}$ is proved in /3/.

Let $\mathbf{a}, \boldsymbol{\lambda}$ be certain continuous functions of the parameter $\alpha \in R^s$.

Assertion. If for $\alpha = \alpha_0$ the vector $\boldsymbol{\lambda}(\alpha_0)$ is non-resonant, then uniformly in $\varphi_0 \in T^{n+1}$ $\{\varphi \bmod 1\}$

$$\lim_{\alpha\to\alpha_0} \mu(\mathbf{a}(\alpha), \boldsymbol{\lambda}(\alpha), \varphi_0) = \mu(\mathbf{a}(\alpha_0), \boldsymbol{\lambda}(\alpha_0))$$

Remarks. 1°. For fixed α the function $\mu(\mathbf{a}, \boldsymbol{\lambda}, \varphi_0)$ is generally discontinuous in φ_0, when $\boldsymbol{\lambda}(\alpha)$ is a resonance vector.

2°. If the function $A(t)$ vanishes for certain values of the time, its argument is not defined. In such a case it is customary to distinguish the "first" and "left" arguments of the function $A(t)$. When passing through a zero of multiplicity p the right argument of the function $A(t)$ receives an increment πp as $t\to\infty$, while the left receives the increment $(-\pi p)$ /4/. The right and left mean motions μ^+ and μ^- /3/ that are in agreement for non-resonance

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sets of frequencies are defined correspondingly. The assertion formulated is valid for both the left and the right mean motions.

Proof of the assertion. Let $\lambda_0(\alpha)\neq 0$. We define $B_{ij_1\ldots j_g}(\alpha)\in T^{n+1}\cap\{\varphi_0=0\}$ as the set of points $\varphi_0=(0,\varphi_{01},\ldots,\varphi_{0n})$ such that the corresponding function $A(t)$ satisfies the following conditions:

1°. $A(t)$ has just j_m zeros of multiplicity m in the half-interval $[0, 1/\lambda_0)$.

2°. Let δ^+ be the number of zeros $\{t_s^+\}$ of the function $\operatorname{Im} A(t)$ such that

$$t_s^+\in[0,1/\lambda_0),\ \operatorname{Im}A(t_s^++0)>0,\ \operatorname{Im}A(t_s^+-0)<0,\ \operatorname{Re}A(t_s^+)<0$$

3°. Let δ^- be the number of zeros $\{t_s^-\}$ of the function $\operatorname{Im}A(t)$ such that

$$t_s^-\in[0,1/\lambda_0),\ \operatorname{Im}A(t_s^-+0)<0,\ \operatorname{Im}A(t_s^--0)>0,\ \operatorname{Re}A(t_s^-)<0$$

Then $i=\delta^--\delta^+$.

It follows from the Bol'-Weyl construction /1, 2/ that for arbitrary initial conditions the mean motion is calculated thus:

$$\mu^\pm(a,\lambda,\varphi_0)=\lim_{N\to\infty}\frac{1}{N}\sum_{m=0}^{N-1}\theta^\pm(\psi_0+m\omega(\alpha),\alpha)$$

$$\theta^\pm(\psi,\alpha)=2\pi\lambda_0(\alpha)\sum_{i,j_1,\ldots,j_g}\Big(i\pm\frac{1}{2}\sum_{m=1}^{g}mj_m\Big)\chi(\psi,B_{ij_1\ldots j_g}(\alpha))$$

$$\omega(\alpha)=\Big(\frac{\lambda_1}{\lambda_0},\ldots,\frac{\lambda_n}{\lambda_0}\Big),\quad \psi_0=\Big(\varphi_{01}-\varphi_{00}\frac{\lambda_1}{\lambda_0},\ldots,\varphi_{0n}-\varphi_{00}\frac{\lambda_n}{\lambda_0}\Big)$$

For any $\varepsilon>0$ there exists a neighbourhood $U_1\in D$ of the point α_0 and two continuous functions $F_1(\psi,\varepsilon)$, $F_2(\psi,\varepsilon)$ on T^n such that

$$F_1(\psi,\varepsilon)\leqslant\theta^\pm(\psi,\alpha)\leqslant F_2(\psi,\varepsilon),\ \forall(\psi,\alpha)\in T^n\times U_1 \tag{1}$$

$$\int_{T^n}[F_2(\psi,\varepsilon)-F_1(\psi,\varepsilon)]\,d\psi\leqslant\varepsilon/3$$

Indeed, if U_2 is a sufficiently small neighbourhood of α_0, then the number of zeros of the analytic functions $\operatorname{Im}A(t)$ in

$$\Big[0,\operatorname{sign}\lambda_0\cdot\sup\Big|\frac{1}{\lambda_0(\alpha)}\Big|\Big)$$

is limited by the constant γ uniformly in $\alpha\in U_2$. The multiplicity of each zero of $A(t)$ is also limited uniformly in $\alpha\in U_2$. Then

$$|\theta^\pm(\psi,\alpha)|\leqslant\Gamma,\quad\forall(\psi,\alpha)\in T^n\times U_2$$

For fixed α the functions $\theta^+(\psi,\alpha)$ and $\theta^-(\psi,\alpha)$ agree everywhere except the set

$$V(\alpha)=\bigcup_i\bigcup_{\Sigma j_m^2\neq 0}B_{ij_1\ldots j_g}(\alpha),\quad 0<\operatorname{mes}_{n-1}V(\alpha)<\infty$$

If $S(\alpha,\varepsilon_1)$ is an ε_1-neighbourhood of the set $V(\alpha)$, then for $\alpha\in U_3$, where U_3 is a sufficiently small neighbourhood of α_0

$$V(\alpha)\in S[(\alpha_0,\varepsilon_1),\quad \varepsilon_1=\frac{\varepsilon}{48\Gamma\operatorname{mes}_{n-1}V(\alpha_0)}$$

$$\sup_{\alpha\in U_3}\lambda_0(\alpha)-\inf_{\alpha\in U_3}\lambda_0(\alpha)\leqslant\frac{\varepsilon}{24\pi\gamma}.$$

Let

$$G^\pm(\psi)=\begin{cases}\pm\Gamma, & \psi\in S(\alpha_0,\varepsilon_1)\\ \sup\limits_{\alpha\in U_2\cap U_3}\Big(\inf\limits_{\alpha\in U_2\cap U_3}\Big)\theta^\pm(\psi,\alpha), & \psi\notin S(\alpha_0,\varepsilon_1)\end{cases}$$

Then

$$\int_{T^n}(G^+(\psi)-G^-(\psi))\,d\psi=\int_{S(\alpha_0,\varepsilon_1)}(G^+-G^-)\,d\psi+\int_{T^n\setminus S(\alpha_0,\varepsilon_1)}(G^+-G^-)\,d\psi\leqslant$$

$$2\Gamma\Big(\frac{\varepsilon}{48\Gamma}+o(\varepsilon)\Big)+2\pi\gamma\big(\sup_{\alpha\in U_2\cap U_3}\lambda_0(\alpha)-\inf_{\alpha\in U_2\cap U_3}\lambda_0(\alpha)\big)<\frac{\varepsilon}{6}$$

Now applying the standard method of approximating step functions by continuous functions to G^- and G^+, we obtain the functions F_1, F_2 that satisfy inequalities (1).

The subsequent reasoning is similar to the reasoning in Kozlov's theorem about time means /5/.

According to Weierstrass's theorem on approximation /6/, trigonometric polynomials $Q_1(\psi)$, $Q_2(\psi)$ exist such that

$$|Q_m(\psi)-F_m(\psi,\varepsilon)|<\varepsilon/6,\quad m=1,2$$

Then the polynomials $P_1(\psi) = Q_1(\psi) - \varepsilon/6$, $P_2(\psi) = Q_2(\psi) + \varepsilon/6$ satisfy the conditions

$$P_1(\psi) < \theta^{\pm}(\psi, \alpha) < P_2(\psi), \quad \forall (\psi, \alpha) = T^n \times U_1 \tag{2}$$
$$\int_{T^n} [P_2(\psi) - P_1(\psi)]\, d\psi < \varepsilon$$

Let $d = \max(\deg P_1, \deg P_2)$. Since $\omega(\alpha_0) > \notin Z$ for $\alpha = \alpha_0 < k$ for all $k \in Z^n$, $k \neq 0$, then in a certain neighbourhood $U_4 \in D$ of the point α_0 the following inequalities hold

$$|\langle k, \omega(\alpha)\rangle - k_0| > \varepsilon > 0, \quad 0 < |k| = \sum_{m=1}^{n} |k_m| \leqslant d \tag{3}$$

If

$$g(\psi) = \sum_{0 \leqslant |k| \leqslant d} g_k \exp[2\pi i \langle k, \psi\rangle]$$

then upon compliance with the inequalities (3) we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} g(\psi_0 + m\omega(\alpha)) = g_0 = \int_{T^n} g(\psi)\, d\psi$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} P_q(\psi_0 + m\omega(\alpha)) = \Lambda_q = \int_{T^n} P_q(\psi)\, d\psi, \; q = 1, 2$$

By virtue of the first inequality in (2)

$$\frac{1}{N} \sum_{m=0}^{N-1} P_1(\psi_0 + m\omega(\alpha)) < \frac{1}{N} \sum_{m=0}^{N-1} \theta^{\pm}(\psi_0 + m\omega(\alpha), \alpha) < \frac{1}{N} \sum_{m=0}^{N-1} P_2(\psi_0 + m\omega(\alpha))$$

and we have in the limit as $N \to \infty$

$$\Lambda_1 \leqslant \mu^{\pm}(a(\alpha), \lambda(\alpha), \varphi_0) \leqslant \Lambda_2, \quad \alpha \in U_1 \cap U_4 = U$$

But by virtue of the first inequality in (2) for $\alpha = \alpha_0$

$$\Lambda_1 \leqslant \mu(a(\alpha_0), \lambda(\alpha_0)) \leqslant \Lambda_2$$

from which by virtue of the second inequality in (2)

$$|\mu^{\pm}(a(\alpha), \lambda(\alpha), \varphi_0) - \mu(a(\alpha_0), \lambda(\alpha_0))| < \varepsilon, \quad \alpha \in U$$

which it was required to prove.

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